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On the Hamiltonian structure of Ermakov systems

F Haas and J Goedert
Instituto de Física, UFRGS, Caixa Postal 15051, 91500-970 Porto Alegre, RS, Brazil

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Abstract. A canonical Hamiltonian formalism is derived for a class of Ermakov systems specified by several different frequency functions. This class of systems comprises all known cases of Hamiltonian Ermakov systems and can always be reduced to quadratures. The Hamiltonian structure is explored to find exact solutions for the Calogero system and for a non-central potential with dynamic symmetry. Some generalizations of these systems possessing exact solutions are also identified and solved.

1. Introduction

Ermakov systems have been intensively studied since the late 1960s both in view of their nice mathematical properties and of the application potential of their celebrated invariants. More recently, the identification of additional structures in the original Ermakov–Pinney system and in its generalized version, the Ermakov–Lewis–Ray–Reid (ELRR) system (see e.g. [1, 2] and references therein for an updated appraisal of the subject and its applications) has called for extra attention. After the early scrutiny of the symmetry properties of the ELRR system [1] and the exploration of several generalization schemes [3, 4], attention has been, more recently, centred on the joint existence of a second independent constant of motion or a Hamiltonian for subclasses of Ermakov systems [5–8]. This particular topic is important *per se* because, for Ermakov systems, the existence of a second constant of motion usually implies, as shown in section 2, complete integration. Most importantly, however, the Hamiltonian structure is fundamental in various contexts of physics such as quantization and perturbation theory.

A generic ELRR system for two independent variables is given by equations (1) and (2) below. This paper considers Ermakov systems for which ω is generalized to be a function of the variables x and y besides the time t. By choosing the functions f and g appropriately, it is possible to fit the equations into a Hamiltonian formalism and eventually derive another constant of motion, distinct from the commonly known Lewis–Ray–Reid invariant (LRRI) given by equation (3). When two integrals exist for this Hamiltonian Ermakov system then we show that the equations are completely integrable.

A Hamiltonian structure for the ELRR system can be enforced by specializing the arbitrary functions f, g and ω. For $w = \omega(t)$, that is, when ω is a function of t only, the Hamiltonian constraint involves only f and g. This specialization has already served two important applications [7] where the potential in the resulting Hamiltonian is essentially built with what remains of the functions f and g. In a more general situation where ω also depends on the dynamic variables, the Hamiltonian constraint is less restrictive and applies to a wider class of admissible systems.

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In this paper we consider the Hamiltonian property of the generalized ELRR system in which ω may depend not only on time but also on x and y. In section 2, a quadratic form in the momenta is proposed for the Hamiltonian and the consequences of this choice are explored analytically. In this way, all known cases of Hamiltonian Ermakov systems in two spatial components are recovered and generalized. The general class of nonlinear systems thus determined is, in addition, shown to be exactly integrable. This remarkable fact encounters applications in several areas of physics. In section 3 we apply the technique in two different situations which, in our understanding, illustrate its applicability both in recovering results already known in the literature and in identifying new exactly solvable models. Among the new results we quote a modified version of the Calogero potential [9] and a variation of a non-central Hartmann potential, known to possess dynamic symmetry [10–12].

The most general system in two configuration variables that qualifies for an Ermakov or ELRR system is usually written as

$$\ddot{x} + \omega^2 x = \frac{1}{yx^2} f(y/x) \tag{1}$$

$$\ddot{y} + \omega^2 y = \frac{1}{xy^2} g(x/y) \tag{2}$$

where the overdot stands for the time derivative, f and g are arbitrary functions of their arguments and ω is an arbitrary function of time t and of x, y and their time derivatives of first and higher orders. For practical reasons, we consider only $\omega = \omega(x, y, t)$.

The system of equations (1) and (2) possesses the Lewis–Ray–Reid invariant [13]

$$I = \tfrac{1}{2}(x\dot{y} - y\dot{x})^2 + \int^{y/x} f(s)\,\mathrm{d}s + \int^{x/y} g(s)\,\mathrm{d}s. \tag{3}$$

The invariant (3) persists for arbitrary dependence of ω on x and y [1, 5]. We can, therefore, merge ω^2 and g in a single function and redefine f according to the following rules:

$$\omega^2 \longmapsto \Omega^2 \equiv \omega^2(x, y, t) - \frac{1}{xy^3} g(x/y) \tag{4}$$

$$f(s) \longmapsto F(s) = f(s) - \frac{1}{s^2} g(1/s) \tag{5}$$

$$g(s) \longmapsto 0. \tag{6}$$

These redefinitions simplify future considerations and cast the ELRR system and the LRRI in the more compact forms

$$\ddot{x} + \Omega^2 x = \frac{1}{yx^2} F(y/x) \tag{7}$$

$$\ddot{y} + \Omega^2 y = 0 \tag{8}$$

$$I = \tfrac{1}{2}(x\dot{y} - y\dot{x})^2 + \int^{y/x} F(s)\,\mathrm{d}s. \tag{9}$$

The transformations (4–6) also indicate that the conventional ELRR system comprises only *two* arbitrary functions and not three as implied by the traditional notation. Of course, all previous results found in the literature, obtained in the standard notation, remain true. Notice, however, that we can always consider the Hamiltonian property of the ELRR system in the form (7) and (8) without any loss of generality.

2. Hamiltonian formalism

As already mentioned, ω in (1) and (2) can be a function of time and of any combinations of the dynamic variables x, y and their time derivatives of arbitrary order. In this work we consider the Hamiltonian property of the ELRR system for which the frequency function is allowed to depend not only on time (as usual) but also on x and y. In this case, the resulting constraint becomes less restrictive and we can find a much wider class of ELRR systems satisfying the Hamiltonian property.

We shall consider for the Hamiltonian of the Ermakov system (7) and (8) the function

$$H = \tfrac{1}{2}Ap_x^2 + Bp_x p_y + \tfrac{1}{2}Cp_y^2 + V(x, y, t) \tag{10}$$

where A, B and C are numbers such that $AC - B^2 \neq 0$ and $V(x, y, t)$ is a potential function depending on time and on the spatial variables.

The ansatz (10) is justified in Douglas theory for two-dimensional Lagrangian systems [14], where it is shown that the coefficients of the quadratic terms in the velocities in a Lagrangian are constants of motion, at least for velocity-free force fields. In particular, these coefficients can be taken as numerical constants. As one can show, the addition of a term linear in the momenta does not alter the generality of the description. Finally, two cases of Hamiltonian Ermakov systems known in the literature are of this proposed form [5, 6].

We now impose that the canonical Hamilton equations generate the ELRR system (7) and (8). The Hamiltonian function (10) generates the following equations for the motion of the system:

$$\dot{x} = Ap_x + Bp_y \tag{11}$$

$$\dot{y} = Bp_x + Cp_y \tag{12}$$

$$\dot{p}_x = -\frac{\partial V}{\partial x} \tag{13}$$

$$\dot{p}_y = -\frac{\partial V}{\partial y}. \tag{14}$$

This first-order system of equations in (x, y, p_x, p_y) can be easily recast in the equivalent second-order system of equations for (x, y),

$$\ddot{x} + \Omega^2 x = -A\frac{\partial V}{\partial x} - B\frac{\partial V}{\partial y} + \Omega^2 x \tag{15}$$

$$\ddot{y} + \Omega^2 y = -B\frac{\partial V}{\partial x} - C\frac{\partial V}{\partial y} + \Omega^2 y \tag{16}$$

where the terms proportional to Ω^2 were conveniently added to both sides. The comparison of equation (16) with (8) leads to the conclusion that the admissible frequencies must satisfy

$$\Omega^2 = \frac{1}{y}\left(B\frac{\partial V}{\partial x} + C\frac{\partial V}{\partial y}\right). \tag{17}$$

Also, the comparison of equation (15) with equation (7), when Ω is given by (17), shows that the potential must obey the linear first-order partial differential equation

$$(Bx - Ay)\frac{\partial V}{\partial x} + (Cx - By)\frac{\partial V}{\partial y} = \frac{1}{x^2}F(y/x). \tag{18}$$

The characteristic equations associated to (18) can be written in the form

$$\frac{\mathrm{d}x}{Bx - Ay} = \frac{\mathrm{d}y}{Cx - By} = \frac{x^2\mathrm{d}V}{F(y/x)} \tag{19}$$

and have—as can be easily checked—the general solution

$$V = \frac{1}{2}\Lambda(q,t) + \frac{1}{q}\int^{s} F(s')\,\mathrm{d}s'. \tag{20}$$

Here $\Lambda(q,t)$ is an arbitrary function of its arguments, and the new variables q and s are defined in terms of the dynamic variables and the parameters of the Hamiltonian as

$$q = Ay^2 - 2Bxy + Cx^2 \tag{21}$$

$$s = y/x. \tag{22}$$

It is also convenient to define the function

$$\xi(s) \equiv As^2 - 2Bs + C \tag{23}$$

so that $q = x^2\xi(s)$.

In the conventional notation, the resulting Hamiltonian ELRR system implied by the admissible frequencies and potentials, now reads

$$\ddot{x} + \rho\frac{\partial\Lambda}{\partial q}x = \frac{1}{yx^2}\bar{f}(y/x) \tag{24}$$

$$\ddot{y} + \rho\frac{\partial\Lambda}{\partial q}y = \frac{1}{xy^2}\bar{g}(x/y) \tag{25}$$

where

$$\bar{f}(s) = 2\rho\frac{s}{\xi^2}\int^{s} F(s')\,\mathrm{d}s' + \frac{s}{\xi}(As - B)F(s) \tag{26}$$

$$\bar{g}(1/s) = 2\rho\frac{s^3}{\xi^2}\int^{s} F(s')\,\mathrm{d}s' + \frac{s^2}{\xi}(Bs - C)F(s) \tag{27}$$

and

$$\rho = AC - B^2. \tag{28}$$

There remain two arbitrary functions in the Hamiltonian ELRR system, namely $\Lambda(q,t)$ and $F(y/x)$. In fact, $\bar{f}$ and $\bar{g}$ in (24) and (25) are determined by the single homogeneous function F, and one can check directly that they satisfy

$$s(Bs - C)\frac{\mathrm{d}\bar{f}}{\mathrm{d}s} + (C + 2Bs)\bar{f} = (As - B)\frac{\mathrm{d}\bar{g}}{\mathrm{d}s} + \left(A + \frac{2B}{s}\right)\bar{g}. \tag{29}$$

It is interesting to compare this result with those in the literature. Cerveró and Lejarreta's Hamiltonian Ermakov systems [6] are obtained from the present formalism by setting $A = C = 1$, $B = 0$, and $\Lambda = \omega^2(t)q$ in (10) and (20). The resulting functions $\bar{f}$ and $\bar{g}$ do satisfy their Hamiltonian constraint [6], which is precisely relation (29) specialized for the appropriate parameter values. This formalism has already been used to study the propagation of elliptic Gaussian beams in nonlinear, dispersive media [7, 8]. Also the completely integrable class of Ermakov systems determined by Goedert [5] is Hamiltonian and derivable from the present formalism. In this case the right choice is $A = C = 0$, $B = 1$, and $\Lambda = 2\int^{-q/2} w^2(q')\,\mathrm{d}q'$. Needless to say, the functions $\bar{f}$ and $\bar{g}$ resulting from this prescription do satisfy the integrability condition stated in [5].

According to the Liouville–Arnold theorem, $2n$-dimensional Hamiltonian systems possessing n independent constants of motion in involution with compact level surfaces are integrable by quadratures [15]. For these systems, the motion is quasiperiodic and restricted to n-dimensional tori. The present class of four-dimensional Hamiltonian ELRR systems will possess two independent constants of motion in involution provided that the

Hamiltonian does not depend on time. In this case, H itself is a constant of the motion independent of the LRRI. So, when the function $\Lambda(q, t)$, which is the source of time-dependence in the Hamiltonian, does not contain t, one can expect that the problem is completely integrable. In fact, the level surfaces $H =$ constant and $I =$ constant are not compact in general, but the differential equations nevertheless are reducible to quadratures. Consequently the Hamiltonian ELRR system treated here must be included in the non-trivial class of solvable ELRR systems, to which belong, for instance, some systems analysed by Govinder and Leach who considered frequency functions depending only on time [16].

The Hamiltonian formalism can be used to solve elegantly the equations of motion. Clearly x and y are not the natural coordinates for the problem. Changing from (x, y) to the new coordinates (q, s), recasts the Hamiltonian and the LRRI in the form

$$H = 2\rho q p_q^2 + \frac{1}{2}\Lambda(q, t) + \frac{I(s, p_s)}{q} \tag{30}$$

$$I = \tfrac{1}{2}\xi^2(s) p_s^2 + \int^s F(s')\, \mathrm{d}s' \tag{31}$$

where p_q and p_s are the momenta conjugate to q and s, respectively. This transformation decouples the dynamics and allows one to treat separately the subsets (p_q, q) and (p_s, s) of the phase space. Moreover, when Λ is time-independent, H is also an invariant. In this situation, we can proceed in a manner similar to that used in the energy integral method of standard classical mechanics and split the problem into two separable ordinary differential equations,

$$(\mathrm{d}q/\mathrm{d}t)^2 = 4\rho(2qH - q\Lambda(q) - 2I) \tag{32}$$

$$(\mathrm{d}s/\mathrm{d}t)^2 = 2q^{-2}(t)\xi^2(s)\left(I - \int^s F(s')\, \mathrm{d}s'\right). \tag{33}$$

Equations (32) and (33) can be successively solved in terms of quadratures yielding a formal solution to the ELRR system. In fact, the solution of (33) requires the knowledge of $q(t)$ obtained from (32).

The structure of (33) suggests the rescaling of the time variable according to

$$\mathrm{d}\tau(t) = \mathrm{d}t/q(t). \tag{34}$$

Such rescaling is applicable globally only when $q(t)$ is positive definite in time and τ is monotonic. Under these circumtances system (32) and (33) reads

$$\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 = 4\rho q^2\, (2qH - q\Lambda(q) - 2I) \tag{35}$$

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2 = 2\xi^2(s)\left(I - \int^s F(s')\, \mathrm{d}s'\right) \tag{36}$$

which is a decoupled set of separable equations and, therefore, reducible to quadratures. This procedure specifies a *general* solution of the problem that involves four arbitrary constants, namely H, I and two constants arising from (35) and (36).

One interesting remark concerning the Hamiltonian Ermakov system is the following: changing $\Lambda(q)$ according to $\Lambda(q) \longmapsto \Lambda(q) + k_1$ or according to $\Lambda(q) \longmapsto \Lambda(q) + 2k_2/q$, where k_1 and k_2 are constants, is equivalent to changing the values of the contants H or I in (32) and, therefore, will not change the nature of the integral to be performed. This transformation can be explored to identify variations of known, exactly solvable, systems that are also exactly solvable. On the one hand, the addition of k_1 to Λ does not lead to a relevant variation of the problem, since it only implies a change in the numerical value of H. On the other hand, the addition of $2k_2/q$ to Λ does imply a *qualitative* change in the potential with no relevant change in the calculations. We only need to replace I in the original equations according to

$$I \longmapsto I + k_2. \tag{37}$$

As a concluding remark to this section, we stress that the Hamiltonian ELRR has been reduced to quadratures. Whether these quadratures can actually be performed globally is a different question to be examined in each particular application. Another important follow-up remark concerns the nonlinear superposition law [13] associated with the ELRR systems. In general, this nonlinear superposition law is implicit in the sense that it cannot be actually applied in view of the coupling between the equations. However, when at least one of the equations decouples, the integration can be carried through and the corresponding nonlinear superposition law becomes explicit. For Hamiltonian ELRR systems, we arrive at equations (34) and (36) which constitute an explicit nonlinear superposition law. This is clear since $s(t)$ is constructed using $q(t)$ obtained from a decoupled equation, namely equation (32).

3. Sample applications

In this section we work out some sample applications of the theory. In particular, we analyse the Calogero potential and a super integrable example of a non-central potential. Several other potentials which represent generalizations of either the harmonic or the Coulomb potentials could be treated in a similar way at least in what concerns the dynamics of the two-dimensional motion obtained by projection in an appropriate plane. The super integrable Hartmann potential and some generalizations or variations of it (see [11, 12]) certainly belong to this category. A generalized version of the coupled Pinney equations of interest in two-layer shallow-water wave theory [3] can also be solved analytically by the formalism of this paper.

3.1. The Calogero system as a Hamiltonian ELRR system

As a first example to illustrate the analytical integration of a Hamiltonian ELRR system, we consider the Calogero potential and its associated system [9], which is a one-dimensional three-body problem given by the Hamiltonian

$$\begin{aligned} H_C = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{\sigma^2}{6}((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2) \\ + \frac{g_1}{(x_1 - x_2)^2} + \frac{g_2}{(x_2 - x_3)^2} + \frac{g_3}{(x_3 - x_1)^2} \end{aligned} \tag{38}$$

where σ, g_1, g_2 and g_3 are non-negative constants. A rescaling of time and space coordinates allows one to set $\sigma \equiv 1$. Moreover, in view of the translational invariance of the problem, we transform to the centre of mass and Jacobi coordinates:

$$\begin{aligned} R &= \tfrac{1}{3}(x_1 + x_2 + x_3) \\ x &= \tfrac{1}{\sqrt{2}}(x_1 - x_2) \\ y &= \tfrac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3). \end{aligned} \tag{39}$$

The centre of mass only executes free motion, and the (x, y) dynamics is described by the reduced Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{g_1}{2x^2} + \frac{2g_2}{(x - \sqrt{3}y)^2} + \frac{2g_3}{(x + \sqrt{3}y)^2}. \tag{40}$$

As can be easily checked, the reduced Hamiltonian (40) is of Ermakov type. Moreover, being autonomous, it is integrable. The coefficients A, B and C in (10) become, in this case, $A = C = 1$ and $B = 0$. Consequently, $q = x^2 + y^2 = r^2$ and the corresponding Ermakov potential reads

$$V = \frac{q}{2} + \frac{1+s^2}{2q}\left(g_1 + \frac{4g_2}{(1-\sqrt{3}s)^2} + \frac{4g_3}{(1+\sqrt{3}s)^2}\right). \tag{41}$$

By comparing this form and equation (20) we find that for the Calogero system

$$\Lambda = q \tag{42}$$

and

$$\int^s F(s')\,\mathrm{d}s' = \frac{1+s^2}{2}\left(g_1 + \frac{4g_2}{(1-\sqrt{3}s)^2} + \frac{4g_3}{(1+\sqrt{3}s)^2}\right). \tag{43}$$

Equation (32) can be solved analytically for Λ given in (42) and it can be verified directly that the rescaling $t \mapsto \tau$ is properly defined, that is, $q(t)$ is positive definite in time. The resulting system of equations to be solved is now

$$\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 = 4q^2(2Hq - q^2 - 2I) \tag{44}$$

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2 = (1+s^2)^2\left(2I - (1+s^2)\left(g_1 + \frac{4g_2}{(1-\sqrt{3}s)^2} + \frac{4g_3}{(1+\sqrt{3}s)^2}\right)\right). \tag{45}$$

It is now easy to find the solution $q(\tau')$, where we make the convenient replacement $\tau \mapsto \tau' \equiv \sqrt{2I}\tau$ (I is strictly positive for the Calogero system),

$$q(\tau') = \frac{2I}{H - \sqrt{H^2 - 2I}\sin(2(\tau' + c_1))} \tag{46}$$

where c_1 is the integration constant arising from (44). This integration constant and the integration constant c_2 arising from the integration of (45) below can always be expressed in terms of the initial position and of H and I.

The function $s(\tau)$ can be evaluated in closed form for a few different values of g_i. We choose $g_1 = g_2 = g_3 \equiv g$, which is perhaps the most interesting case yielding solutions in terms of circular functions,

$$s(\tau') = \tan\left(\frac{1}{3}\sin^{-1}\left(\left(1 - \frac{9g}{2I}\right)^{1/2}\sin(3(\tau' + c_2))\right)\right) \tag{47}$$

where c_2 is the integration constant referred to before.

The last two equations express the parametric orbits of the problem and are a general solution involving four integration constants, namely H, I, c_1 and c_2. This solution is in full agreement with the results of Khandekar and Lawande [17], but are obtained in a more systematic way.

As mentioned before, the addition of $2g_4/q$ to Λ, where g_4 is a constant, does not alter the nature of the analytic solutions. It only requires the change $I \to I + g_4$ in all formulae. This proves the existence of closed form solutions for the modified Hamiltonian:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\sigma^2}{2}(x^2 + y^2) + \frac{g_1}{2x^2} + \frac{2g_2}{(x - \sqrt{3}y)^2} + \frac{2g_3}{(x + \sqrt{3}y)^2} + \frac{g_4}{x^2 + y^2}. \tag{48}$$

Therefore, in the original variables, the one-dimensional three-body problem described by the Hamiltonian

$$\bar{H}_{\mathrm{C}} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{\sigma^2}{6}\left((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\right) + \frac{g_1}{(x_1 - x_2)^2}$$

$$+\frac{g_2}{(x_2-x_3)^2}+\frac{g_3}{(x_3-x_1)^2}+\frac{3g_4}{(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2} \qquad (49)$$

is also exactly solvable. $\bar{H}_C$ can be viewed as an integrable modification of the Calogero system. Notice that the term in g_4 is a new contribution that did not belong to the original system. We remark that this generalization is possible thanks to the fact that, in Jacobi coordinates, the Calogero system possesses the structure of a Hamiltonian ELRR system.

3.2. A Hamiltonian with dynamic symmetry

The non-central problem described by the Hamiltonian

$$H=\frac{1}{2}(p_x^2+p_y^2)-\frac{\sigma}{\sqrt{x^2+y^2}}+\frac{g_1}{y^2}+\frac{g_2 x}{y^2\sqrt{x^2+y^2}} \qquad (50)$$

where σ, g_1 and g_2 are positive constants, is known to possess a dynamic symmetry group [10] and is separable in parabolic and polar coordinates. Three-dimensional extensions of H are super-integrable systems [11], and have received attention as a non-central force problem amenable to Feynman quantization [12]. Several other two-dimensional versions of three-dimensional super-integrable models can be reduced to Hamiltonian ELRR systems. We selected (50) as a good example to illustrate the technique proposed in section 2.

A rescaling of time and space allows one to set $\sigma\equiv 2$ without any loss of generality. For this problem also $q=x^2+y^2=r^2$ and it is convenient to introduce polar coordinates so that $s=\tan\theta$. The potential, in (r,θ) coordinates, now reads

$$V=-\frac{2}{r}+\frac{1}{r^2\sin^2\theta}(g_1+g_2\cos\theta). \qquad (51)$$

Comparison of equations (51) and (20) shows that in this case

$$\Lambda(q)=-\frac{4}{r} \qquad (52)$$

and

$$\int^{s=\tan\theta}F(s')\,\mathrm{d}s'=\frac{1}{\sin^2\theta}(g_1+g_2\cos\theta). \qquad (53)$$

Let us restrict considerations to the cases where the rescaled time τ is well defined and monotonic. A detailed analysis shows that for positive definite LRRI invariant (sufficiently high angular momentum) the trajectories never cross the origin. For $I>0$ the variable $q=r^2$ never vanishes and τ is monotonically increasing as can be calculated directly from equation (34). We note that for $I>0$, $q\neq 0$ is also required for the right-hand side of equation (32) to be positive definite, a necessary condition for the existence of real valued solutions. We therefore consider $I>0$ and reduce the original problem to the set of differential equations

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2=2r^2(Hr^2+2r-I) \qquad (54)$$

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2=2\left(I-\frac{1}{\sin^2\theta}(g_1+g_2\cos\theta)\right). \qquad (55)$$

A direct integration of (54) and (55) (with $\tau\mapsto\tau'\equiv\sqrt{2I}\tau$) yields the parametric orbit equation

$$r(\tau')=\frac{I}{1-\sqrt{1+HI}\sin(\tau'+c_1)} \qquad (56)$$

$$\cos\theta(\tau') = -\frac{g_2}{2I}\left(1 - \sqrt{1 + 4I(I - g_1)/g_2^2}\,\sin(\tau' + c_2)\right) \tag{57}$$

where c_1, c_2 are integration constants which can be expressed in terms of the initial position and of H and I. For negative values of the energy, the motion is bounded. When the energy is positive we find open trajectories that escape to infinity for finite values of τ'. However, in the original parameter t this process is regular and takes an infinite amount of time as expected on physical grounds. Another interesting feature of the bounded motion is the fact that it does not explore the entire range $\theta = 0$ to $\theta = 2\pi$. This is evident from equation (57), which inplies $\chi_- \leqslant \cos\theta \leqslant \chi_+$, where

$$\chi_\mp = -\frac{g_2}{2I}\left(1 \pm \sqrt{1 + 4I(I - g_1)/g_2^2}\right). \tag{58}$$

It is easy to verify that $\chi_+ < 1$ which corresponds to the fact that the trajectories never visit that sector of the plane where $\theta \leqslant \arccos\chi_+$. It is also clear that for sufficiently small values of g_2 there may exist another excluded sector around $\theta = \pi$. This is the case when $\chi_- > -1$ which is possible if and only if $2I > g_2$ and $g_1 > g_2$.

As in the first example, the addition of a term $2g_3/q$ to $\Lambda(q)$, where g_3 is a new constant, does not alter the calculations. The new exactly solvable potential is given by

$$V = \frac{1}{2}(p_x^2 + p_y^2) - \frac{\sigma}{\sqrt{x^2 + y^2}} + \frac{g_1}{y^2} + \frac{g_2 x}{y^2\sqrt{x^2 + y^2}} + \frac{g_3}{x^2 + y^2}. \tag{59}$$

Here we should stress that the term proportional to g_3 is novel and represents an incorporation to the original potential that preserves the integrability property. Again this was possible thanks to the Hamiltonian character of this Ermakov system.

4. Conclusions

A quite general class of exactly integrable Hamiltonian ELRR systems has been identified and solved. The basic result comes from the fact that an Ermakov system being Hamiltonian (with a quadratic form in the momenta) is exactly solvable in terms of quadratures. Also the ultimate equations for these systems are decoupled, a fact that leads to practical nonlinear superposition laws.

Another important feature of the formalism stems from the fact that for each exactly solvable model there always exists a modified version of the problem that is also integrable analytically. This provides a mechanism to spawn new integrable Ermakov systems. Two examples of physically interesting applications were treated in detail to illustrate the practical value of the method. The scope of the applications of the method, however, is much wider and a detailed assessment of its reach is an open question that deserves further study.

Some open questions are readily identified. A first open question concerns the expansion of the method to higher-dimensional Ermakov systems. From the point of view of physics, this would be most interesting, mainly for three space dimensions. A second open question concerns the Hamiltonian character of ELRR systems with a velocity-dependent frequency function. Still another interesting question concerns the perturbation theory of ELRR systems. This completely unexplored subject has not been touched so far, perhaps because of the lack of an Hamiltonian structure. The results of this paper open a new prospective for such issues.

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